Gauge theories on quantum spaces: Finiteness, integrability, instability

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Outline

General ideas - 1

- Noncommutative Geometry founded by A. Connes $\sim$ 1980.
- Provides noncommutative generalizations of concepts of usual topology, differential geometry, index theory,...
- Key: Set-up a duality between a "space" and (associative) algebra to provide algebraic description of topological, metric, differential,... properties of the ”space”.
- Commutative example for such duality:
  - Gel'fand-Naimark duality between commutative C*-algebras and (locally compact) spaces
  - Commutative space $\mathcal{M} \longleftrightarrow$ algebra of functions on $\mathcal{M}$.

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General ideas - 2

- When space no longer commutative :
  Noncommutative algebra $\longrightarrow$ Noncommutative space
  i.e view noncommutative algebra as a model for a noncommutative space whenever this latter has no pictorial representation. Algebraic description of concepts from topology, geometry, ... so that:
  objects of commutative world have noncommutative analogs.
- Correspondance / Dictionnary: Examples
  – points $\longrightarrow$ (pure) states
  – Vector bundles over spaces $\longrightarrow$ modules over algebras ,
  – K-theory and its dual K-homology $\longrightarrow$ K-theory of C*-algebras,
  – de Rham cohomology $\longrightarrow$ cyclic homology,
  – Riemann manifold $\longrightarrow$ spectral triple ("Dirac operator")
  – Riemann metric $\longrightarrow$ spectral distance ("Dirac operator")
  – Atiyah Singer theorem $\longrightarrow$ Connes-Moscovici Index Theorem

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Useful examples for physics

- Activity in mathematics +++
- Correspondance / Dictionnary: Other useful examples
  – de Rham complex $\longrightarrow$ derivation-based differential calculus
  – vector fields $\longrightarrow$ derivations,
  – connexion and curvature have generalizations ($\ni$ usual Koszul connection acting on sections of vector bundles),...
- $Q : \mathbb{A} = (\mathcal{F}, \star)$ (deformation) $\rightarrow \hat{\mathbb{A}} = (Q(\mathcal{F}), .) \subset \mathcal{L}(\mathcal{H})$
- Q: quantization map, (invertible) *-algebra morphism, $f \star g = Q^{-1}(Q(f).Q(g))$, $\star$: deformation of usual product
- Examples: Weyl map (Moyal product).

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Derivations of an associative algebra

- Derivation of $\mathbb{A}$: $X : \mathbb{A} \to \mathbb{A}$, $X(ab) = X(a)b + aX(b)$, $\forall a, b \in \mathbb{A}$
- Der$\mathbb{A}$ has canonical Lie algebra and $\mathcal{Z}(\mathbb{A})$-module structures: $[X, Y](a) \equiv X(Y(a)) - Y(X(a))$, $(zX)(a) = z(X(a))$, $\forall z \in \mathcal{Z}(\mathbb{A})$
- Derivation-based differential calculus:(Sketch of construction)
- follows rather closely construction of ordinary (de Rham) forms
- Start from Lie subalgebra, $\mathcal{Z}(\mathbb{A})$-submodule of Der$\mathbb{A}$, says $\mathcal{G}$
- 0-forms: $\mathbb{A}$. Define differential $d : \Omega^n_{\mathcal{G}} \to \Omega^{n+1}_{\mathcal{G}}$, such that $d^2 = 0$.
  $d\omega_0(X) = X(\omega_0)$, $\forall \omega_0 \in \mathbb{A}$,
  $d\omega_1(X, Y) = X(\omega_1(Y)) - Y(\omega_1(X)) - \omega_1([X, Y])$, $X, Y \in \mathcal{G}$,
  ω_1 $\mathcal{Z}(\mathbb{A})$-linear maps from $\mathcal{G}$ to $\mathbb{A}$.
  Can be extended to n-forms,$\mathcal{Z}(\mathbb{A})$-n-linear antisymmetric maps from $\mathcal{G}$ to $\mathbb{A}$. $d^2 = 0$ thanks to structures of $\mathcal{G}$.

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Noncommutative geometry in physics

- NCG involves interesting features for (quantum) physics:
  - quantum physics is NC; what matters are observables (algebra) not points
  - NCG can be used to deal with observables related to space-time,
  - NC coordinates $[x_\mu, x_\nu] \neq 0$ "solve" usual objections of existence of continuous space-time and commuting coordinates at Planck scale. $\longrightarrow$Starting point of Noncommutative Field Theories (NCFT).
- Notice: Not only for Planck scale. Recent reformulations of Standard Model with NCG: Higgs mass consistent with data.
- NCG: possible way to investigate "quantum gravity". 2 others: i) String, ii) Loop gravity and related

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Noncommutative Fields Theories

- Non local field theories with complicated kinetic operator
- can be represented as a "matrix" model
- sometimes coupling between IR and UV scales (UV/IR mixing)
- vacuum quantum instabilities
- "Modern formulation":
  - 1986 Witten string field theory
  - $\sim$ 1990 Fuzzy sphere NCFT, κ-Minkowski
  - 1992 Yang-Mills Higgs model from matrix geometry
  - 1998 NCFT on $\mathbb{R}^4_\theta$ as some low energy regime of string
  - from 1998: $\uparrow\uparrow$ interest: renormalisability properties, matrix model formulation of NCFT, investigations of gauge theories
  - 2004 Exactly solvable scalar model on $\mathbb{R}^4_\theta$
  - 2004 : All order renormalisable φ^4 models on $\mathbb{R}^4_\theta$
  - $\uparrow\uparrow$ literature on NCFT: $\mathbb{R}^4_\theta$, κ-Minkowski, $\mathbb{R}^3_\theta$, NC Tori,...
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Connections and curvatures - basics

- Pick a right-module $\mathbb{M}$ on $\mathbb{A}$
- Connection $\nabla_X : \mathbb{M} \to \mathbb{M}$, $\nabla_X(ma) = \nabla_X(m)a + mX(a)$ (+...)
- Curvature: $F_{(X,Y)}(m) \equiv [\nabla_X, \nabla_Y](m) - \nabla_{[X,Y]}(m)$
- Group of gauge transformation: automorphisms of $\mathbb{M}$ (compatible with some hermitean structure further assumed) $\nabla_X^g = g^\dagger \nabla_X g,\ F_{X,Y} = g^\dagger F_{X,Y} g$
- Assume now $\mathbb{M} = \mathbb{C} \otimes \mathbb{A}$ (plus $h(a_1, a_2) = a_1^\dagger a_2$). Then, hermitean connection entirely determined by $\nabla_X(\mathbb{I})$. 1-form connection defined by $A : X \to A(X) := \nabla_X(\mathbb{I})$.

$$A^g(X) = g^\dagger A(X) g + g^\dagger X(g),\ F^g_{(X,Y)} = g^\dagger F_{(X,Y)} g$$

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- Connection $\nabla_X : \mathbb{M} \to \mathbb{M}$, $\nabla_X(ma) = \nabla_X(m)a + mX(a)$ (+...)
- Curvature: $F_{(X,Y)}(m) \equiv [\nabla_X, \nabla_Y](m) - \nabla_{[X,Y]}(m)$
- Group of gauge transformation: automorphisms of $\mathbb{M}$ (compatible with some hermitean structure further assumed) $\nabla^g_X = g^\dagger \nabla_X g,\ F_{X,Y} = g^\dagger F_{X,Y} g$
- Assume now $\mathbb{M} = \mathbb{C} \otimes \mathbb{A}$ (plus $h(a_1, a_2) = a_1^\dagger a_2$). Then, hermitean connection entirely determined by $\nabla_X(\mathbb{I})$. 1-form connection defined by $A : X \to A(X) := \nabla_X(\mathbb{I})$.

$$A^g(X) = g^\dagger A(X) g + g^\dagger X(g),\ F^g_{(X,Y)} = g^\dagger F_{(X,Y)} g$$

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Gauge-invariant connection and covariant coordinates

- Assume there exists some $\eta(X)$ such that $X(a) = [\eta(X), a]$
- $\eta(X)$ defines a connection which is gauge-invariant:

$$\nabla_X^{inv}(a) = X(a) - \eta(X)a$$

- Natural covariant object:

$$\mathcal{A}(X) = \nabla_X^{inv} - \nabla_X = A(X) + \eta(X)$$

- Gauge transformations:

$$(\nabla^{inv})_X^g = \nabla_X^{inv},\ \mathcal{A}^g(X) = g^\dagger \mathcal{A}(X) g$$

- $A(X)$: gauge potential - $\mathcal{A}(X)$ covariant tensor form (covariant coordinates)

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Noncommutative gauge theory on $\mathbb{R}^3_\lambda$

- $\mathbb{R}^3_\lambda = (\mathcal{F}(\mathbb{R}^3), \star)$, $f \star g = \exp(\delta_{\mu\nu} x_0 + i\varepsilon_{\mu\nu\rho} x_\rho \partial_{u_\mu} \partial_{v_\nu}) f(u) g(v)_{|u,v=0}$ with $x_0^2 = \sum_{\mu=1}^3 x_\mu^2$ (no $\star$). Not the most tractable presentation.
- Here, convenient to represent it as an algebra of operators. One finds:

$$\mathbb{R}^3_\lambda = \oplus_{j \in \frac{\mathbb{N}}{2}} \mathbb{M}_{2j+1}(\mathbb{C})$$

- Relations $[x_\mu, x_\nu] = i\lambda\varepsilon_{\mu\nu\rho} x_k,\ x_0^2 + \lambda x_0 = \sum_{\mu=1}^3 x_i^2$, Center $\mathcal{Z} = \mathbb{C}[C_2 = \sum_{\mu=1}^3 x_\mu^2]$, x_0 "radius" of fuzzy spheres $\mathbb{M}_{2j+1}$.
- Notice: $\mathbb{R}^3_\lambda$ is the convolution algebra of $SU(2)$. Interesting presentation to deal with group C*-algebra version of $\mathbb{R}^3_\lambda$.
- Differential calculus $\mathcal{G} = \{D_\mu = i[\theta_\mu, .]\}$, $\mu = 1, 2, 3$. Set $A(D_\mu) := A_\mu$. 1-form gauge-invariant connection defined by

$$\theta_\mu = \frac{x_\mu}{\lambda^2}$$

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Family of harmonic gauge theory on $\mathbb{R}^3_\lambda$

- Set $\Phi_\mu := \mathcal{A}_\mu$. $\Phi_\mu^g = g^\dagger \Phi_\mu g$.
- Observe: $(\theta_\mu\theta^\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda) \Rightarrow Tr\big((\theta_\mu\theta^\mu)\Phi_\nu\Phi^\nu\big)$ gauge invariant
  Not true for $\mathbb{R}^4_\theta$! Harmonic term allowed by gauge invariance.
- Look for (positive) gauge invariant $S(\Phi_\mu)$ with $\Phi_\mu = 0$ as minimum and harmonic term.
- One obtains (in the gauge $\Phi_3 = \theta_3$)

$$S = \frac{2}{g^2} Tr\big(\Phi Q \Phi^\dagger + \Phi^\dagger Q \Phi\big) + \frac{16}{g^2} Tr\big((\Omega+1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega-1)\Phi\Phi^\dagger\Phi^\dagger\Phi\big)$$

- $\Phi = \Phi_1 + i\Phi_2$. Kinetic self-adjoint operator is

$$Q = M + \mu x^2 + 8\Omega L(\theta_3^2) + i4(\Omega-1)L(\theta_3)D_3$$

for $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$.

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Properties of the harmonic gauge theories

- S describes dynamics of fluctuations of Φ_μ around $\Phi_\mu = 0$ or alternatively fluctuations of the gauge potential A_μ around the gauge invariant connection.
- Convenient to use canonical orthogonal basis of $\mathbb{R}^3_\lambda$. Built from canonical basis of each $\mathbb{M}_{2j+1}(\mathbb{C})$, v^j_{mn}, $-j \le m, n \le j$.

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Noncommutative product becomes a "matrix product". $S(\Phi)$ can be viewed as a matrix model.

Theorem (Géré, Jurić, JCW)

The amplitudes of the (ribbon) diagrams for any of the gauge theories described by $S(\Phi)$ with $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$, are finite to all orders in perturbation.

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- Finiteness origin:
  - Sufficient decay for propagator as $j \to \infty$ (UV regime)
  - j plays role of natural UV cut-off (kind of "external moment")
  - Existence of (finite) upper bound for general amplitude $\mathfrak{A}^j_{\mathcal{D}}$

Lemma (Géré, Jurić, JCW)

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq K \frac{j^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I}$$

- Comments: Ribbon diagram $\mathcal{D} : (V, I, F, B)$. Can be mapped on Riemann surface of genus g with $2 - 2g = V - I + F$.
- Power counting: RHS finite when $\omega(\mathcal{D}) = I + 2B + 2(2g - 2) + V \geq 0$.
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  - Sufficient decay for propagator as $j \to \infty$ (UV regime)
  - j plays role of natural UV cut-off (kind of "external moment")
  - Existence of (finite) upper bound for general amplitude $\mathfrak{A}^j_{\mathcal{D}}$

Lemma (Géré, Jurić, JCW)

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq K \frac{j^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I}$$

- Comments: Ribbon diagram $\mathcal{D} : (V, I, F, B)$. Can be mapped on Riemann surface of genus g with $2 - 2g = V - I + F$.
- Power counting: RHS finite when $\omega(\mathcal{D}) = I + 2B + 2(2g - 2) + V \geq 0$. $g \geq 1$ true. $g = 0$ compute diagrams : found to be finite.

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Other gauge theories on $\mathbb{R}^3_\lambda$ and $\mathbb{R}^{2n}_\theta$

- Gauge theory on $\mathbb{R}^3_\lambda$ $\rightarrow$ fluctuations of gauge potential A_μ around $A_\mu = 0$ in Géré, Vitale, JCW, Phys. Rev. D 90, 045019 (2014). **One-loop non-zero tadpole diagram $\Rightarrow$ vacuum instability** Getting Propagator: Dual Hahn polynomials and Favard theorem (Géré, JCW, J.Phys.:Conf. Ser.634(2015) 012006).
- Gauge theories on $\mathbb{R}^{2n}_\theta$, $n = 1, 2$: still problem with UV/IR mixing (except for $n = 1$) and determination of vacua.
  Moyal space: Informally the algebra "generated" by
  $$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$
  Candidate action: de Goursac, JCW, Wulkenhaar, Eur.Phys.J.C51(2007)977
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Gauge theories on $\mathbb{R}^3_\lambda$: Exact formulas at $\Omega = \frac{1}{3}$

- Assume $\Omega = \frac{1}{3}$. Interaction → only $Tr(\Phi\Phi^\dagger\Phi\Phi^\dagger)$.

$$Z(Q) = \prod_{j \in \frac{\mathbb{N}}{2}} \left(\int D\Phi^j D\Phi^{j\dagger} \exp[Kin + tr_j(\Phi^j\Phi^{j\dagger}\Phi^j\Phi^{j\dagger})] \right) \qquad (1)$$

$\Phi = \sum_j \sum_{m,n} \Phi^j_{mn} v^j_{mn}$, $(\Phi^j)_{mn} = \Phi^j_{mn}$, $\Phi^j \in \mathbb{M}_{2j+1(\mathbb{C})}$

- Use:
Singular value decomposition: $\Phi^j = U^\dagger R^j V$, R^j positive diagonal matrix. → Decouples radial (R^j) from angular (U, V) part.
Harish-Chandra measure formula to integrate over angular part.
Radial part is simple.

$$\int [DU]\, e^{z\, \mathrm{tr}(MUNU^\dagger)} = \frac{1}{\Delta(M)\Delta(N)} \prod_{k=1}^{n-1} k!\, z^{\frac{n(1-n)}{2}} \det_{1\le k,l\le n} \left(e^{z\lambda^M_k \lambda^N_l} \right),$$

$\forall z \in \mathbb{C}\backslash\{0\}$, $[DU]$ Haar measure on $U(n)$

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Exact formula 1

- $$Z_j(Q) = (N^j(g^2)\,(2j+1)!)\frac{\det_{-j\le m,n\le j}\left(f(\omega^j_m+\omega^j_n)\right)}{\Delta^2(Q^j)} \quad ,$$
  $$f(x) = \sqrt{\frac{\pi g^2}{128(j+1)}}\ \mathrm{erfc}(x\sqrt{\frac{(j+1)}{64g^2}})\ e^{x\frac{(j+1)}{64g^2}}$$
  $\Delta(Q^j)$ Vandermonde determinant of kinetic operator Q^j with eigenvalues ω^j_m.
- Ratio of determinant $\rightarrow$ τ-function for integrable hierarchy.

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Concluding comments

- Each Z_j related to 2D Toda hierarchy (truncation of the gauge theory to a single fuzzy sphere)
- Z related to an infinite tower of Toda hierarchies
- Gauge theory is at least "formally" exactly solvable
- Free energy? 2-point function? (T. Jurić, JCW in progress).
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